Complex geometry of conifolds
and 5-brane wrapped on 2-sphere

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Abstract

We investigate solutions of type II supergravity which have the product $\mathbb{R}^{1,3} \times M^6$ structure with non-compact M^6 factor and which preserve at least four supersymmetries. In particular, we consider various conifolds and the $N = 1$ supersymmetric "NS5-brane wrapped on 2-sphere" solution recently discussed in hep-th/0008001. In all of these cases, we explicitly construct the complex structures, and the Kähler and parallel (3,0) forms of the corresponding M^6. In addition, we verify that the above solutions preserve respectively eight and four supersymmetries of the underlying type II theory. We also demonstrate that the ordinary and fractional D3-brane (5-brane wrapped on 2-cycle) solutions on singular, resolved and deformed conifolds, and the (S-dual of) NS5-brane wrapped on 2-sphere can be obtained as special cases from a universal ansatz for the supergravity fields, i.e. from a single 1-d action governing their radial evolution. We show that like the 3-branes on conifolds, the NS5-brane on 2-sphere background can be found as a solution of first order system following from a superpotential.

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1 Introduction

Supergravity solutions that preserve some of the supersymmetry of underlying theory have found many applications in the exploration of perturbative and non-perturbative properties of string theory. An important example is the AdS/CFT correspondence which asserts that string theory on the $AdS_5 \times S^5$ background is related to $N = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions [1]. Recently, attempts have been made to extend this correspondence to find string theory duals of $N = 1$ supersymmetric gauge theories (see, in particular, [2–5] and also [6–9]).

A remarkable observation made in [5] is that a non-trivial supersymmetric solution constructed in [10] may be interpreted as describing a near-throat region of a large number of NS5-branes wrapped on 2-sphere. In what follows we refer to this solution as "NS5-branes wrapped on 2-sphere" and denote it as NS5$_{S^2}$. The S-dual of NS5$_{S^2}$ represents D5-branes wrapped on S^2, and thus string theory on this background was conjectured to be dual to pure $N = 1$ SYM theory in four dimensions.

One of the aims of the present paper is to investigate the geometrical properties, and clarify furher the residual supersymmetry, of the ten-dimensional NS⊗NS background of [5, 10]. In addition, the holomorphic geometry of singular, deformed and resolved Calabi-Yau conifolds will be explored.

The solutions that we shall consider belong to the class of ten-dimensional NS⊗NS backgrounds which are products $M^{10} = M^4 \times M^6$ of a four- and a six-dimensional space with possibly a warp factor multiplying the metric of M_4. In addition, they (i) preserve at least four of the original supersymmetries, and (ii) have non-trivial dilaton Φ and Kalb-Ramond field strength H.[1]

An important simplification in investigating NS⊗NS backgrounds is that some of the Killing spinor equations are parallel transport equations of a connection with torsion (see, e.g., Section 2). Since for the applications mentioned above $M^4 = \mathbb{R}^{1,3}$, the M^6 part of spacetime has properties similar to those of Calabi-Yau spaces. In fact, the Killing spinor equations impose strong restrictions on the existence of such solutions [12]. In particular, it was shown in [13] that if one assumes that M^6 is compact and the dilaton is a globally defined function on it, then the only such spaces that preserve at least four supersymmetries are the Calabi-Yau spaces with $H = 0$ and $\Phi = \text{const}$.

To have supersymmetric solutions with *running* dilaton, one has to consider backgrounds for which M^6 is *non-compact*. The solution of [5, 10], i.e. NS5$_{S^2}$ background, provides an example. The metric on the spacetime $M^{10} = \mathbb{R}^{1,3} \times M^6$ is the direct sum of the flat metric on $\mathbb{R}^{1,3}$ and the NS5$_{S^2}$ metric on M^6. It turns out that this six-space has many similarities with non-compact Calabi-Yau manifolds like, for example, the conifold and its resolved and deformed versions [14] (see also [15, 16]). These conifolds appeared as transverse spaces in the (ordinary and fractional) D3-brane supergravity solutions constructed in [3, 4, 8, 17].

The expressions for the metrics of the spaces that we shall investigate are known, but it is not so for their basic geometrical properties as *complex six-dimensional* manifolds. In particular, complex structures, and Kähler and parallel (3,0)-forms (see Section 2 for

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[1]Non-constant dilaton is of course expected to be related to running gauge coupling on gauge theory side. Such backgrounds may be of interest in the context of the general framework suggested in [11].

the details) have not been explicitly determined. Both the complex structures and the (3,0)-forms are important for a study of *string theory* in these backgrounds. For example, complex structures are used to construct the ($N = 2$) supersymmetry transformations for the string world-sheet action, while the (3,0)-forms are associated with conservation laws.[2] Other applications of the complex structures and parallel (3,0)-forms are in the context of geometry and, in particular, in the investigation of calibrated submanifolds. These, in turn, have applications in the context of branes.

Let us summarize the contents of this paper. We shall explore below the complex-geometric properties of singular, deformed and resolved conifolds as well as of NS5$_{S^2}$ space, presenting their complex structures, Kähler forms and parallel (3,0)-forms of these manifolds. For the conifolds the complex structures are apparent in the holomorphic coordinate system that was initially used for the construction of their metrics. In Section 3 we shall give the expressions for their complex structures in terms of another coordinate system which is also suitable for deriving the parallel (3,0)-forms. In Section 4 we shall turn to the case of NS5$_{S^2}$ space and derive the corresponding complex structure, Kähler form and the parallel (3,0)-form. We will also give the expression for the associated holomorphic (3,0)-form. In this way we will explicitly verify that the NS5$_{S^2}$ solution preserves four of the supersymmetries of the type II supergravity theory, in agreement with the arguments presented in [5, 10].

It has been shown in [3, 8] that the solutions representing configurations of (ordinary and fractional) D3-branes on singular [2, 3], deformed [4] and resolved [8] conifolds can be obtained by solving a system of first order equations. The latter may be derived as BPS type of equations from a one-dimensional action admitting a superpotential.. As we shall demonstrate in Section 5, the NS5$_{S^2}$ background [5, 10] is also a solution of a a collection of first order equations which arises as a BPS system for a one-dimensional action with a superpotential. This gives another indication of a similarity between the D3-brane on conifold solutions and the NS5$_{S^2}$ background. In addition, this provides an independent indication that the NS5$_{S^2}$ background is supersymmetric. In the process, we shall explain how all of the solutions of [3–5, 8] can be obtained from a single "interpolating" ansatz for the ten-dimensional metric, dilaton and p-form fields.

Appendix A contains some technical details of determining the complex structure of the NS5$_{S^2}$ space. In Appendix B supergravity backgrounds which have the same symmetry as that of Calabi spaces are examined. It is found that the only such solutions in the NS⊗NS are the Calabi metrics.

2 Supersymmetry and NS⊗NS solutions

The type II supergravity field equations that involve the metric g, NS⊗NS three-form field strength H and the dilaton Φ are (in the string frame)

$$R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + 2\nabla_m \partial_n \Phi = 0 , \qquad (2.1)$$

$$\nabla_p \left(e^{-2\Phi} H^{pmn} \right) = 0 , \qquad (2.2)$$

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[2]The latter, however, can develop anomalies at the quantum level [18].

where ∇ is the Levi-Civita connection of the metric g and $m, n, \ldots = 0, \ldots, 9$. The dilaton equation $\nabla^2 e^{-2\Phi} = \frac{1}{6} e^{-2\Phi} H^{mnk} H_{mnk}$ follows from the above two assuming that the central charge integration constant vanishes.

We shall consider solutions of the type $\mathbb{R}^{1,3} \times M^6$ which preserve at least four supersymmetries of the ten-dimensional theory. We shall assume that H, Φ denote the restriction of the fields to M^6, i.e. H is a closed 3-form and Φ is a scalar on the M^6, and the non-trivial part of the ten-dimensional spacetime metric is that of M^6. We shall use $M, N, K, .. = 1, 2, ..., 6$ for the indices of M^6.

For the background to preserve four supersymmetries, the following conditions are required (see [12, 19]):

(i) M^6 should be a Hermitian manifold with respect to a complex structure J which is constructed from the Killing spinors.

(ii) One of the connections with torsion $\nabla^{\pm} = \nabla \pm \frac{1}{2}H$, say ∇^+, should have its holonomy contained in $SU(3)$, i.e.

$$\nabla_M^+ J^N{}_P = 0 , \qquad R_{MN}^+{}^P{}_Q J^Q{}_P = 0 , \qquad (2.3)$$

where R^+ is the curvature of the ∇^+ connection.

(iii) The Kähler form Ω of J should satisfy

$$\frac{1}{2}\Omega^{MN} H_{MNR} = -2 J^K{}_R \partial_K \Phi . \qquad (2.4)$$

This equation arises from the dilatino Killing spinor equation.

In [19] it was shown that if J is integrable, then the first equation (2.3) is equivalent to

$$d\Omega + i_J H = 0 , \qquad (2.5)$$

where i_J is the inner derivation with respect to J, i.e.

$$i_J H_{MNP} = -3 J^K{}_{[M} H_{|K|NP]} . \qquad (2.6)$$

Because for the backgrounds we are considering the holonomy of ∇^+ is contained in $SU(3)$, the manifold M^6 admits a (3,0)-form $\tilde{\eta}$ which is parallel, i.e. covariantly constant with respect to the connection ∇^+. Conversely, suppose that M^6 is a hermitian manifold that admits a (3,0)-form. Now if both the complex structure and (3,0)-form are parallel with respect to ∇^+ connection, then the holonomy of ∇^+ is contained in $SU(3)$ and therefore M^6 admits parallel spinors. So such background will be supersymmetric provided that some of the parallel spinors satisfy the dilatino Killing spinor equation as well.

Apart from the parallel (3,0)-form $\tilde{\eta}$, M^6 admits in addition a holomorphic (3,0)-form η given by

$$\eta = e^{-2\Phi}\tilde{\eta} . \qquad (2.7)$$

The existence of η has been used to show [13] that there are no solutions of this type with *compact* manifolds M^6 for which H is non-vanishing and the dilaton is a globally defined scalar on the manifold. The proof given in [13] can also be extended, after imposing appropriate boundary conditions, to *non-compact* manifolds M^6. So the backgrounds for

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which ∇^+ has holonomy contained in $SU(n)$, H is non-vanishing and Φ is non-constant are severely restricted. Some examples in $4k$ dimensions which are not products have been given in [20].

It was recently argued in [5] that the NS5$_{S^2}$ solution [10] provides another example of NS$\otimes$NS background with non-compact M^6 which preserves four supersymmetries and so the holonomy of ∇^+ is $SU(3)$. We shall establish this explicitly in Section 4.

A special case arises when H vanishes. Then the dilaton Φ must be constant, the manifold M^6 is Kähler and the holonomy of the Levi-Civita connection ∇ is contained in $SU(3)$, i.e. M^6 is a Calabi-Yau manifold. The existence of Calabi-Yau metrics for compact and non-compact manifolds has been established using powerful analytical methods. For compact Kähler manifolds, the necessary and sufficient condition required is the triviality of the canonical bundle. For the *non-compact* case, additional information regarding boundary conditions is necessary.

Nevertheless, only very few non-trivial (non direct-product) examples of such metrics are known explicitly, like the Calabi metrics on the resolved $\mathbb{C}^n/\mathbb{Z}_n$ singularity [21] and the metrics on (singular, deformed and resolved) conifolds in [14, 15] and [8]. Before examining the NS5$_{S^2}$ background, we shall begin in Section 3 with the conifolds and construct explicitly the corresponding complex structures and holomorphic (3,0)-forms. In this way we shall confirm that the conifolds is a class of supersymmetric backgrounds satisfying the requirements (i)-(iii) above with vanishing torsion and constant dilaton.

Unlike the NS5$_{S^2}$ solution, the conifolds preserve eight supersymmetries in the context of type II theories. This is because for the conifolds $\nabla = \nabla^+ = \nabla^-$ since the torsion vanishes. Thus the holonomy of both ∇^+ and ∇^- connections is contained is $SU(3)$. In contrast, for the NS5$_{S^2}$ background although the holonomy of ∇^+ is $SU(3)$, the holonomy of ∇^- is $SO(6)$, and, as a consequence, there are no parallel spinors with respect to ∇^-. Thus the NS5$_{S^2}$ solution preserves half the number of supersymmetries compared to the conifolds.

Before we proceed with the case by case investigation, we conclude with some general remarks regarding the parameterization of almost complex structures and (3,0)-forms on six-dimensional manifolds. The generalization to higher dimensions is straightforward. At every point of the six-dimensional manifold, the almost complex structures that are compatible with a given metric are parameterized by the coset space[3] $SO(6)/U(3)$. Given a metric, it is expected that the associated Kähler form of the almost complex structure is locally parameterized by six independent functions. Now, given a metric and an almost complex structure on M^6, at every point of the manifold the compatible (3,0) forms are parameterized by the coset $U(3)/SU(3)$. So it is expected that given a frame of (1,0)-forms associated with the almost complex structure, the (3,0)-form is determined up to a phase.

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[3]Note that $SO(6)/U(3) = \mathbb{C}P^3$ and its dimension is the same as that of the manifold. We thank Stefan Ivanov for pointing this out to us.

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3 Complex structure of 6-d metrics on conifolds

3.1 The singular conifold

Kähler structure

The singular conifold is the complex three-dimensional subspace in $\mathbb{C}^4$ defined by the equation [14]

$$\sum_{A=1}^{4} (w^A)^2 = 0 \ , \tag{3.1}$$

where $\{w^A; A = 1, 2, 3, 4\}$ are the coordinates of $\mathbb{C}^4$. Clearly, such a space is smooth everywhere apart from singular point $w^A = 0$. Let us review the construction of the Kähler metric on the conifold following [14, 15]. Observe that (3.1) can be rewritten as

$$\det W = 0 \ , \qquad W = w^i \sigma_i + w^4 \mathbf{1} \ , \tag{3.2}$$

where $\{\sigma_i; i = 1, 2, 3\}$ are the Pauli matrices, $\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i\epsilon_{ijk}\sigma_k$. Let us define a "radial" coordinate

$$\rho^2 = \text{Tr}(WW^\dagger) \ , \tag{3.3}$$

and a function $K = K(\rho^2)$ which will be identified with the Kähler potential. Then the Kähler metric on the conifold is

$$\begin{aligned}
ds^2 &= \partial_\alpha \partial_{\bar\beta} K \ dz^\alpha d\bar{z}^{\bar\beta} \\
&= K'' |\text{Tr}(W^\dagger dW)|^2 + K' \text{Tr}(dW dW^\dagger) \ ,
\end{aligned} \tag{3.4}$$

where $(...)' = \frac{d(...)}{d\rho^2}$ and $\{z^\alpha; \alpha = 1, 2, 3\}$ are some complex coordinates on the conifold. The metric can be expressed in terms of five angular coordinates and the radial coordinate ρ by first parameterizing W as

$$W = \rho \ L_1 Z^{(0)} L_2^\dagger \ , \tag{3.5}$$

where the 2×2 matrices $(L_1, L_2) \in SU(2) \times SU(2)$ represent the five angular coordinates, and

$$Z^{(0)} = \frac{1}{2}(\sigma_1 + i\sigma_2) \ . \tag{3.6}$$

To determine the coordinate dependence of L_1, L_2, it is convenient to introduce an additional coordinate so that the six coordinates can be identified with the Euler angles on $SU(2) \times SU(2)$. Then one can introduce the left invariant one-forms $\{e^i, \tilde{e}^i\}$ $(i = 1, 2, 3)$ on $SU(2) \times SU(2)$ group and write

$$L_1^\dagger dL_1 = \frac{1}{2} i e^j \sigma_j \ , \qquad L_2^\dagger dL_2 = \frac{1}{2} i \tilde{e}^j \sigma_j \ , \tag{3.7}$$

so that the Maurer-Cartan equations are

$$de^i = \frac{1}{2} \epsilon^i{}_{jk} e^j \wedge e^k \ , \qquad d\tilde{e}^i = \frac{1}{2} \epsilon^i{}_{jk} \tilde{e}^j \wedge \tilde{e}^k \ . \tag{3.8}$$

The computation of the metric and the Kähler form proceeds by using the properties of the left-invariant forms and by treating the two of the six angular coordinates, which correspond to the "third" Euler angles, as independent, though the final result will effectively depend only on their sum ψ (see [15]). As a result, one finds the metric

$$
\begin{aligned}
ds^2 &= (\rho^2 K'' + K')d\rho^2 + \frac{1}{4}(\rho^4 K'' + \rho^2 K')(e^3 + \tilde{e}^3)^2 \\
&+ \frac{1}{4}K'\rho^2[(e^1)^2 + (e^2)^2 + (\tilde{e}^1)^2 + (\tilde{e}^2)^2] \,.
\end{aligned}
\tag{3.9}
$$

The frames that appear in the expression for the metric are

$$
\begin{aligned}
e^1 &= \sin\frac{\psi}{2}\sin\theta_1 d\phi_1 + \cos\frac{\psi}{2}d\theta_1 \,, \\
e^2 &= -\cos\frac{\psi}{2}\sin\theta_1 d\phi_1 + \sin\frac{\psi}{2}d\theta_1 \,, \\
\tilde{e}^1 &= \sin\frac{\psi}{2}\sin\theta_2 d\phi_2 + \cos\frac{\psi}{2}d\theta_2 \,, \\
\tilde{e}^2 &= -\cos\frac{\psi}{2}\sin\theta_2 d\phi_2 + \sin\frac{\psi}{2}d\theta_2 \,, \\
e^3 + \tilde{e}^3 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \,.
\end{aligned}
\tag{3.10}
$$

The Kähler form of the conifold is defined as

$$
\Omega = \frac{i}{2}\partial\bar{\partial}K = i\left[\text{Tr}(W^\dagger dW) \wedge \rho d\rho K'' + \frac{1}{2}\text{Tr}(dW \wedge dW^\dagger)K'\right] \,.
\tag{3.11}
$$

Using the above relations, Ω can be expressed in terms of the coordinates as

$$
\Omega = \frac{1}{4}\left[d(\rho^2 K') \wedge (e^3 + \tilde{e}^3) + \rho^2 K'(e^1 \wedge e^2 + \tilde{e}^1 \wedge \tilde{e}^2)\right].
\tag{3.12}
$$

From the expressions (3.9),(3.12) it is straightforward to find the adapted frame for this Kähler structure, i.e. the frame in which both the metric and Ω take the standard forms with *constant* coefficients

$$
\begin{aligned}
\text{E}^1 &= \frac{1}{2}(\rho^2 K')^{\frac{1}{2}}e^1, \quad \text{E}^2 = \frac{1}{2}(\rho^2 K')^{\frac{1}{2}}e^2, \quad \text{E}^3 = \frac{1}{2}(\rho^2 K')^{\frac{1}{2}}\tilde{e}^1, \quad \text{E}^4 = \frac{1}{2}(\rho^2 K')^{\frac{1}{2}}\tilde{e}^2, \\
\text{E}^5 &= (\rho^4 K'' + \rho^2 K')^{\frac{1}{2}}\rho^{-1}d\rho \,, \quad \text{E}^6 = \frac{1}{2}(\rho^4 K'' + \rho^2 K')^{\frac{1}{2}}(e^3 + \tilde{e}^3) \,.
\end{aligned}
\tag{3.13}
$$

In particular, the complex structure in this frame is

$$
\begin{aligned}
J(\text{E}^1) &= -\text{E}^2, \quad J(\text{E}^3) = -\text{E}^4, \quad J(\text{E}^5) = -\text{E}^6, \\
J(\text{E}^2) &= \text{E}^1, \quad J(\text{E}^4) = \text{E}^3, \quad J(\text{E}^6) = \text{E}^5 \,.
\end{aligned}
\tag{3.14}
$$

It can be verified by a straightforward computation that the above complex structure is integrable as it is expected. The most convenient way to show this is by using the Frobenius theorem; see Section 4 for details.

<u>Calabi-Yau structure</u>

The above class of Kähler metrics on the conifold contains a Calabi-Yau one, i.e. a metric that it is Ricci flat and so it has holonomy $SU(3)$. One way to find it is by constructing the parallel (3,0)-form η associated with such manifolds as has been explained in Section 2. Indeed, we take

$$\eta = (\mathrm{E}^1 + i\mathrm{E}^2) \wedge (\mathrm{E}^3 + i\mathrm{E}^4) \wedge (\mathrm{E}^5 + i\mathrm{E}^6) \ . \tag{3.15}$$

It turns out that there is no need to add a phase in the (1,0) frame $\{\mathrm{E}^1 + i\mathrm{E}^2, \mathrm{E}^3 + i\mathrm{E}^4, \mathrm{E}^5 + i\mathrm{E}^6\}$ we have chosen.[4] This is clearly a (3,0)-form. It turns out that it is sufficient to check that it is closed as well. This is the case provided

$$\rho^2 G' - G = 0 \ , \tag{3.16}$$

where

$$G \equiv \rho^2 K'(\rho^4 K'' + \rho^2 K')^{\frac{1}{2}} \ . \tag{3.17}$$

Eq. (3.16) can be integrated to give

$$\frac{G}{\rho^2} = \lambda = \mathrm{const} \ , \tag{3.18}$$

where λ is a positive constant; for $\lambda = 0$ the metric is degenerate. Integrating (3.18), one finds

$$K' = \left(\frac{3\lambda^2}{2\rho^2} + \frac{c}{\rho^6}\right)^{\frac{1}{3}} \ , \tag{3.19}$$

where c is another integration constant. There is no need to integrate once more to determine K since the metric, the complex structure and the (3,0)-form all depend on the derivatives of the Kähler potential. Only one combination of λ and c is a nontrivial parameter, so the Kähler potential (3.19) thus determines a *one parameter* family of Calabi-Yau metrics on the conifold.

For $c = 0$, the Kähler potential is

$$K = \left(\tfrac{9\lambda}{4}\right)^{\frac{2}{3}} \rho^{\frac{4}{3}} \ . \tag{3.20}$$

After a redefinition of the ρ coordinate (or setting $\lambda^2 = 2/3$), (3.9) becomes the standard Calabi-Yau metric on the cone over the homogeneous space $T^{(1,1)}$, i.e. a coset $[SU(2) \times SU(2)]/U(1)$ [22]. This 6-d metric is singular at the apex $\rho = 0$ of the cone. It is natural to restrict the parameter c to positive values so that the metric remains regular for all values of ρ^2 apart from $\rho^2 \to 0$.

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[4]However a phase would have been necessary if another (1,0) frame was chosen instead.

3.2 The resolved conifold

<u>Kähler structure</u>

Singularities like that at the apex of the singular conifold in the previous section can be removed either by blowing up the singular point or by deforming the associated set of algebraic equations. The former case will be investigated in this subsection while the latter will be examined in the next one.

The discussion of the complex structure of 6-d metrics on resolved conifold [14] is parallel to the one in the singular conifold case above. The Kähler potential is[5]

$$K = F(\rho^2) + 4a^2\ln(1 + |\Lambda|^2) , \tag{3.21}$$

where

$$\rho^2 = (1 + |\Lambda|^2)(|U|^2 + |Y|^2) . \tag{3.22}$$

Here (U, Y, Λ) are the holomorphic coordinates which can be parameterized in terms of the Euler angles as

$$U = \rho e^{\frac{i}{2}(\psi+\phi_1+\phi_2)} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} , \quad Y = \rho e^{\frac{i}{2}(\psi-\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} , \quad \Lambda = e^{-i\phi_2}\tan\frac{\theta_2}{2} .$$

The metric of the resolved conifold can then be written as [8] (cf. (3.9))

$$\begin{aligned}ds^2 &= (\rho^2 F'' + F')d\rho^2 + \frac{1}{4}(\rho^4 F'' + \rho^2 F')(e^3 + \tilde{e}^3)^2 \\ &+ \frac{1}{4}F'\rho^2[(e^1)^2 + (e^2)^2] + \frac{1}{4}(F'\rho^2 + 4a^2)[(\tilde{e}^1)^2 + (\tilde{e}^2)^2] ,\end{aligned} \tag{3.23}$$

where the 1-forms are expressed in terms of the Euler angles as in (3.10). This metric has explicit $SU(2) \times SU(2)$ invariance and becomes the same as in (3.9) when $a = 0$ (and thus $K = F$).[6] The dependence on the resolution parameter a here is explicit, but after imposing additional conditions, like Ricci-flatness, F may start depending on a.

The Kähler form of the resolved conifold is then (cf. (3.12))

$$\Omega = \frac{i}{2}\partial\bar{\partial}K = \frac{1}{4}\left[d(\rho^2 F') \wedge (e^3 + \tilde{e}^3) + \rho^2 F'e^1 \wedge e^2 + (\rho^2 F' + 4a^2)\tilde{e}^1 \wedge \tilde{e}^2\right]. \tag{3.24}$$

The adapted frame for the metric and the Kähler form is (cf. (3.13))

$$\begin{aligned}E^1 &= \frac{1}{2}(\rho^2 F')^{\frac{1}{2}}e^1 , & E^2 &= \frac{1}{2}(\rho^2 F')^{\frac{1}{2}}e^2 , \\ E^3 &= \frac{1}{2}(\rho^2 F' + 4a^2)^{\frac{1}{2}}\tilde{e}^1 , & E^4 &= \frac{1}{2}(\rho^2 F' + 4a^2)^{\frac{1}{2}}\tilde{e}^2 , \\ E^5 &= (\rho^4 F'' + \rho^2 F')^{\frac{1}{2}}\rho^{-1}d\rho , & E^6 &= \frac{1}{2}(\rho^4 F'' + \rho^2 F')^{\frac{1}{2}}(e^3 + \tilde{e}^3) .\end{aligned} \tag{3.25}$$

The complex structure in this frame takes the same form as in (3.14) and is integrable.

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[5]Here we use ρ instead of r in [8] for the radial coordinate.

[6]For small ρ the S^3 (ψ, θ_1, ϕ_1) part of the metric shrinks to zero size while the S^2 $(\tilde{\epsilon}_2, \phi_2)$ part stays finite with radius a.

<u>Calabi-Yau structure</u>

To find the Calabi-Yau representative in the class of metrics (3.23) we again construct the holomorphic (3,0)-form η. The required form has the same structure as in (3.15)

$$\eta = (\mathrm{E}^1 + i\mathrm{E}^2) \wedge (\mathrm{E}^3 + i\mathrm{E}^4) \wedge (\mathrm{E}^5 + i\mathrm{E}^6) \ . \tag{3.26}$$

It is closed, provided

$$\rho^2 G' - G = 0 \ , \tag{3.27}$$

where (cf. (3.17))

$$G \equiv [\rho^2 F'(\rho^2 F' + 4a^2)(\rho^4 F'' + \rho^2 F')]^{\frac{1}{2}} \ . \tag{3.28}$$

Thus, as in (3.18), $\frac{G}{\rho^2} = \lambda = $ const. Integrating once more, we get

$$(\rho^2 F')^3 + 6a^2(\rho^2 F')^2 = \frac{3}{2}\lambda^2 \rho^4 + c \ , \tag{3.29}$$

which reduces to (3.19) for $a = 0$. This is the same as the equation on F found in [8] from the Ricci-flatness condition (in [8] the constants were chosen to be $\lambda^2 = 2/3$ and $c = 0$). Thus, in general, we get a one (non-trivial) parameter family of Calabi-Yau metrics on the resolved conifold.

3.3   The deformed conifold

<u>Kähler structure</u>

In the deformed conifold case [14] the singular point $w^A = 0$ is excluded by replacing (3.1) by

$$\det W = -\frac{\epsilon^2}{2} \ , \tag{3.30}$$

where ϵ is a "deformation" parameter. Again, one parameterizes W as [15]

$$W = \ \rho \ L_1 Z_\epsilon^{(0)} L_2^\dagger \ , \tag{3.31}$$

where now (cf. (3.5))

$$Z_\epsilon^{(0)} = \frac{1}{2}\alpha(\rho)(\sigma_1 + i\sigma_2) + \frac{1}{2}\beta(\rho)(\sigma_1 - i\sigma_2) \ , \tag{3.32}$$

$$\alpha \equiv \frac{1}{2}\left(\sqrt{1 + \frac{\epsilon^2}{\rho^2}} + \sqrt{1 - \frac{\epsilon^2}{\rho^2}}\right) \ , \qquad \beta \equiv \sqrt{1 - \alpha^2} = \frac{\epsilon^2}{2\rho^2}\alpha^{-1} \ . \tag{3.33}$$

Observe that $\alpha^2 + \beta^2 = 1$. The metric and Kähler form are determined by W through the general expressions (3.4) and (3.11). To compute them, we again organize the angular

variables in terms of the two sets of $SU(2)$ Euler angles and left-invariant forms $\{e^i, \tilde{e}^i\}$ as in the singular conifold case. Then the metric of the deformed conifold

$$
\begin{aligned}
ds^2 = \quad & K''\rho^2 d\rho^2 + K'(d\rho^2 + \rho^2 d\alpha^2 + \rho^2 d\beta^2) \\
+ \quad & \frac{1}{4}[\rho^4(\alpha^2 - \beta^2)K'' + \rho^2 K'](e^3 + \tilde{e}^3)^2 \\
+ \quad & \frac{1}{4}K'\rho^2\Big[(\alpha e^1 - \beta\tilde{e}^1)^2 + (\beta e^1 - \alpha\tilde{e}^1)^2 + (\alpha e^2 + \beta\tilde{e}^2)^2 + (\beta e^2 + \alpha\tilde{e}^2)^2\Big] ,
\end{aligned}
\tag{3.34}
$$

or, equivalently (cf. (3.9),(3.23))

$$
ds^2 = [K''(1 - \frac{\epsilon^4}{\rho^4})\rho^4 + K'\rho^2]\Big[\frac{d\rho^2}{\rho^2(1 - \frac{\epsilon^4}{\rho^4})} + \frac{1}{4}(e^3 + \tilde{e}^3)^2\Big]
$$

$$
+ \frac{1}{4}K'\rho^2\Big[(\alpha e^1 - \beta\tilde{e}^1)^2 + (\beta e^1 - \alpha\tilde{e}^1)^2 + (\alpha e^2 + \beta\tilde{e}^2)^2 + (\beta e^2 + \alpha\tilde{e}^2)^2\Big] .
\tag{3.35}
$$

Similarly, the Kähler form of the deformed conifold is

$$
\Omega = \frac{1}{4}\Big[d[\rho^2 K'(\alpha^2 - \beta^2)] \wedge (e^3 + \tilde{e}^3) + \rho^2 K'(\alpha^2 - \beta^2)(e^1 \wedge e^2 + \tilde{e}^1 \wedge \tilde{e}^2)\Big] .
\tag{3.36}
$$

These expressions reduce to the ones in the singular conifold case in the $\epsilon = 0$ limit.

The adapted frame of this Kähler structure where the metric and the Kähler form have constant components is the following (cf. (3.13),(3.25)):

$$
\begin{aligned}
\mathrm{E}^1 &= \frac{1}{2}(K'\rho^2)^{\frac{1}{2}}(\alpha e^1 - \beta\tilde{e}^1) , & \mathrm{E}^2 &= \frac{1}{2}(K'\rho^2)^{\frac{1}{2}}(\alpha e^2 + \beta\tilde{e}^2) , \\
\mathrm{E}^3 &= \frac{1}{2}(K'\rho^2)^{\frac{1}{2}}(-\beta e^1 + \alpha\tilde{e}^1) , & \mathrm{E}^4 &= \frac{1}{2}(K'\rho^2)^{\frac{1}{2}}(\beta e^2 + \alpha\tilde{e}^2) , \\
\mathrm{E}^5 &= [K''(1 - \frac{\epsilon^4}{\rho^4})\rho^4 + K'\rho^2]^{\frac{1}{2}}\frac{d\rho}{\rho(1 - \frac{\epsilon^4}{\rho^4})^{\frac{1}{2}}} , \\
\mathrm{E}^6 &= \frac{1}{2}[K''(1 - \frac{\epsilon^4}{\rho^4})\rho^4 + K'\rho^2]^{\frac{1}{2}}(e_3 + \tilde{e}_3) .
\end{aligned}
\tag{3.37}
$$

Indeed, like in the singular and resolved conifold cases, in this adapted frame the Kähler form becomes simply

$$
\Omega = \mathrm{E}^1 \wedge \mathrm{E}^2 + \mathrm{E}^3 \wedge \mathrm{E}^4 + \mathrm{E}^5 \wedge \mathrm{E}^6 .
\tag{3.38}
$$

The complex structure is thus

$$
\begin{aligned}
J(\mathrm{E}^1) &= -\mathrm{E}^2 , & J(\mathrm{E}^3) &= -\mathrm{E}^4 , & J(\mathrm{E}^5) &= -\mathrm{E}^6 , \\
J(\mathrm{E}^2) &= \mathrm{E}^1 , & J(\mathrm{E}^4) &= \mathrm{E}^3 , & J(\mathrm{E}^6) &= \mathrm{E}^5 .
\end{aligned}
\tag{3.39}
$$

Calabi-Yau structure

The class of Kähler metrics on the deformed conifold includes also a Calabi-Yau metric. As in the case of the singular conifold (3.15), one can define the (3,0)-form by

$$
\eta = (\mathrm{E}^1 + i\mathrm{E}^2) \wedge (\mathrm{E}^3 + i\mathrm{E}^4) \wedge (\mathrm{E}^5 + i\mathrm{E}^6) .
\tag{3.40}
$$

For the Kähler structure to be associated with the Calabi-Yau metric, the form η must be closed. This is the case provided that (cf. (3.16)–(3.18))

$$\frac{\alpha\beta^2}{\alpha'(\alpha^2 - \beta^2)}G' - G = 0 , \quad \text{i.e.} \quad \rho^2 G' - G = 0 , \tag{3.41}$$

$$G \equiv \rho^2 K'\left[(1 - \frac{\epsilon^4}{\rho^4})\rho^4 K'' + \rho^2 K'\right]^{\frac{1}{2}} . \tag{3.42}$$

We have used the expressions (3.33) for α, β (3.41) to show that the equation for G is the same as in (3.16),(3.27). Integrating it, we find again

$$\frac{G}{\rho^2} = \lambda = \text{const} , \tag{3.43}$$

where λ is assumed to be positive. Integrating this equation once more, we get

$$K' = \frac{1}{\sqrt{\rho^4 - \epsilon^4}}\left[\frac{3}{2}\lambda^2\rho^2\sqrt{\rho^4 - \epsilon^4} + c - \frac{3}{2}\lambda^2\epsilon^4 \log(\rho^2 + \sqrt{\rho^4 - \epsilon^4})\right]^{\frac{1}{3}} , \tag{3.44}$$

where c is another integration constant (again, only one combination of c and λ is a non-trivial parameter). This expression reduces to (3.19) in the $\epsilon \to 0$ limit. As in the conifold case, finding the derivative K' of the Kähler potential with respect to ρ^2 suffices to determine the metric, complex structure and the holomorphic (3,0)-form. The Kähler potential following from (3.44) defines a one-parameter family of Calabi-Yau metrics on the deformed conifold.

4 The Complex Geometry of NS5$_{S^2}$ background

4.1 The solution

The NS⊗NS solution of [10] lifted to ten dimensions has form $\mathbb{R}^{1,3} \times M^6$, where M^6 has topology $\mathbb{R} \times S^2 \times S^3$. The string frame metric is the direct sum of the $\mathbb{R}^{1,3}$ and M^6 metrics. The metric on M^6, the 3-form and the dilaton are given by [5]

$$ds^2 = dr^2 + e^{2g(r)}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{1}{4}\sum_{i=1}^{3}(\epsilon^i - A^i)^2 , \tag{4.1}$$

$$H = -\frac{1}{4}(\epsilon^1 - A^1) \wedge (\epsilon^2 - A^2) \wedge (\epsilon^3 - A^3) + \frac{1}{4}\sum_{i=1}^{3} F^i \wedge (\epsilon^i - A^i) , \tag{4.2}$$

$$e^{2\Phi} = e^{2\Phi_0}\frac{2e^g}{\sinh 2r} , \tag{4.3}$$

respectively, where

$$A^1 = a(r)d\theta , \quad A^2 = a(r)\sin\theta d\varphi , \quad A^3 = \cos\theta d\varphi , \tag{4.4}$$

and

$$a = \frac{2r}{\sinh 2r} , \quad e^{2g} = r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4} . \tag{4.5}$$

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F is the curvature of the connection A, i.e. $F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk}A^j \wedge A^k$, and ϵ^i are the left-invariant one-forms on S^3 which now satisfy (cf. (3.10),(3.8))

$$d\epsilon^i = -\frac{1}{2}\epsilon^i{}_{jk}\epsilon^j \wedge \epsilon^k . \tag{4.6}$$

It is convenient to define the frame in which the metric is diagonal

$$\xi^0 = dr , \quad \xi^1 = e^g d\theta , \quad \xi^2 = e^g \sin\theta d\varphi , \quad \rho^i = \frac{1}{2}(\epsilon^i - A^i) . \tag{4.7}$$

The 3-form H then becomes

$$H = -2\rho^1 \wedge \rho^2 \wedge \rho^3 + \frac{1}{2}\sum_{i=1}^{3} F^i \rho^i . \tag{4.8}$$

In addition, we have

$$A^1 = ae^{-g}\xi^1, \qquad A^2 = ae^{-g}\xi^2, \qquad A^3 = e^{-g}\cot\theta\xi^2 , \tag{4.9}$$

and $F^{1,2} = e^{-g}da \wedge \xi^{1,2}$, $F^3 = (a^2 - 1)e^{-2g}\xi^1 \wedge \xi^2$. Observe also that

$$d\xi^1 = dg \wedge \xi^1 , \qquad d\xi^2 = dg \wedge \xi^2 + e^{-g}\cot\theta\,\xi^1 \wedge \xi^2 , \tag{4.10}$$

$$d\rho^i = -\epsilon^i{}_{jk}\rho^j \wedge \rho^k - \epsilon^i{}_{jk}A^j\rho^k - \frac{1}{2}F^i . \tag{4.11}$$

4.2 Conditions on the Kähler form

To find the complex structure associated with the the $\mathrm{NS5}_{S^2}$ manifold, we consider the most general candidate for a Kähler two-form

$$\Omega = \lambda_i\xi^0 \wedge \rho^i + w_a dr \wedge \xi^a + z_{ia}\rho^i \wedge e^a + \frac{1}{2}\mu^i\epsilon_{ijk}\rho^j \wedge \rho^k + \frac{1}{2}P\epsilon_{ab}\xi^a \wedge \xi^b , \tag{4.12}$$

where $i, j = 1, 2, 3$ and $a, b = 1, 2$ are frame indices raised and lowered with flat Euclidean metric. The components $(\lambda_i, w_a, z_{ia}, \mu^i, P)$ of the form Ω are allowed to depend on all of the coordinates of the manifold M^6.

First, let us find the conditions on Ω such that it can be identified with the Kähler form of an almost complex structure J. For this $J^A{}_B = \delta^{AC}\Omega_{CB}$ should satisfy $J^2 = -1$ ($A, B, C = 1, \ldots, 6$ are frame indices). It is best to perform the calculation by taking (cf. (4.12))

$$\begin{aligned}
i_J(\xi^0) &= -\lambda_i\rho^i - w_a\xi^a , \\
i_J(\rho^i) &= \lambda^i\xi^0 - z^i{}_a\xi^a - \mu^j\epsilon_j{}^i{}_k\rho^k , \\
i_J(\xi^a) &= w^a\xi^0 + z_i{}^a\rho^i - P\epsilon^a{}_b\xi^b ,
\end{aligned} \tag{4.13}$$

and then verifying that $i_J i_J = -1$. Acting on dr twice with i_J, we get the following conditions

$$\begin{aligned}
\lambda_i\lambda^i + w_a w^a &= 1 , \\
\lambda_i z^i{}_a + P w_b \epsilon^b{}_a &= 0 , \\
-\lambda_i\mu^j\epsilon_j{}^i{}_k + z_k{}^a w_a &= 0 .
\end{aligned} \tag{4.14}$$

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Similarly, from $i_J(i_J(\rho^i)) = -\rho^i$, we get

$$
\begin{aligned}
\lambda_i \lambda_j + z_{ia} z_j{}^a + \mu^k \mu_k \delta_{ij} - \mu_i \mu_j &= \delta_{ij} \ , \\
-\lambda_i w_a + P z_{ib} \epsilon^b{}_a + \mu^k \epsilon_k{}^{ij} z_{ja} &= 0 \ .
\end{aligned}
\tag{4.15}
$$

Finally, acting on ξ^a, we find

$$
w_a w_b + z_{ia} z^i{}_b + P^2 \delta_{ab} = \delta_{ab} \ .
\tag{4.16}
$$

General solution of the algebraic constraints (4.14),(4.15),(4.16) is given in Appendix A.

It remains to find the differential equations on the components of Ω (4.12) such that the non-trivial conditions in (2.4) and (2.5) are satisfied. Eq. (2.4) leads to

$$
\begin{aligned}
z_{11} + z_{22} &= 0 \ , \\
-4\mu_1 + \partial_r a e^{-g} w_1 &= -4\lambda_1 \partial_r \Phi \ , \\
-4\mu_2 + \partial_r a e^{-g} w_2 &= -4\lambda_2 \partial_r \Phi \ , \\
-4\mu_3 + (a^2 - 1) e^{-2g} P &= -4\lambda_3 \partial_r \Phi \ , \\
(a^2 - 1) e^{-2g} z_{32} + \partial_r a e^{-g} \lambda_1 &= +4 w_1 \partial_r \Phi \ , \\
(a^2 - 1) e^{-2g} z_{31} - \partial_r a e^{-g} \lambda_2 &= -4 w_2 \partial_r \Phi \ ,
\end{aligned}
\tag{4.17}
$$

where z_{11}, z_{22}, z_{31} and z_{32} are the components of z_{ia} in (4.12). Evaluating $d\Omega$ and $i_J H$ we find from (2.5) the following additional conditions

$$
\begin{aligned}
-D_a \lambda_i + D_i w_a + \lambda_k \epsilon^k{}_{ji} A^j{}_a - e^{-g} \partial_r (z_{ia} e^g) - \mu^k \epsilon_{kji} F^j{}_{ra} & \\
- \frac{1}{2} F^i_{rb} P \epsilon^b{}_a + \frac{1}{2} F^i_{ba} w^b &= 0 \ , \\
-(D_j \lambda_k - (k, j)) + \partial_r \mu_i \epsilon^i{}_{jk} + \frac{1}{2}(z_j{}^a F^k{}_{ra} - (k, j)) &= 0 \ , \\
\lambda_3 F^3_{ab} - (D_a w_b - D_b w_a) - \frac{1}{2} w_2 \cot \theta e^{-g} \epsilon_{ab} & \\
+ \partial_r P \epsilon_{ab} + P \partial_r g \epsilon_{ab} - (F^i_{ra} z_{ib} - (b, a)) &= 0 \ , \\
D_i P \epsilon_{ab} + (D_a z_{ib} - z_{kb} \epsilon^k{}_{ji} A^j{}_a - (b, a)) + z_{i2} \cot \theta e^{-g} \epsilon_{ab} & \\
+ \mu^k \epsilon_{k3i} F^3_{ab} - \frac{1}{2}(F^i_{ra} w_b - (b, a)) &= 0 \ , \\
(D_j z_{ka} - (k, j)) + D_a \mu_i \epsilon_{ijk} & \\
+ \left(-\mu_j A^k_a + \frac{1}{2} F^k_{ra} \lambda_j - \frac{1}{2} F^k_{ba} z_j{}^b - (j, k) \right) &= 0 \ ,
\end{aligned}
\tag{4.18}
$$

where the derivatives D_a and D_i are defined by the equation $df = dr \partial_r f + e^a D_a f + \rho^i D_i f$ for any function f on M^6.

We shall now solve the equations (4.14)–(4.16), (4.17) and (4.18) by assuming that the components of the Kähler form Ω depend only on the *radial coordinate r*. Using this assumption, from (4.18) one finds that

$$
\begin{aligned}
\lambda_1 = \lambda_2 = w_2 = z_{32} = z_{11} = z_{22} = \mu_1 = \mu_2 &= 0 \ , \\
z_{21} + z_{12} &= 0 \ .
\end{aligned}
\tag{4.19}
$$

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From the last two equations in (4.17), we find also that

$$w_2 = z_{31} = 0 \ . \tag{4.20}$$

The condition (4.14) implies that $\lambda_3 = \pm 1$. We shall choose in what follows $\lambda_3 = 1$. Setting $z_{12} = -z_{21} \equiv X$, the equations (4.14)–(4.16) imply that

$$\mu_3 = -P \ , \qquad\qquad X^2 + P^2 = 1 \ . \tag{4.21}$$

The remaining equations in (4.17) and (4.18) give an over-determined linear system on P and X. In particular, P can be found in terms of a, g and Φ from the third equation in (4.17). Simplifying the rest of the equations using $XX' + PP' = 0$ (which follows from (4.21)), the final set of equations is the following:

$$P = -\frac{4\Phi'}{4 + (a^2 - 1)e^{-2g}} \ , \tag{4.22}$$

$$X = \frac{P'}{a'e^{-g}} \ , \tag{4.23}$$

$$X' = -a'e^{-g}P \ , \tag{4.24}$$

$$\frac{1}{2}(a^2 - 1)e^{-2g} - \frac{1}{2}a'e^{-g}X + Pg' = 0 \ , \tag{4.25}$$

$$ae^{-g} + \frac{1}{2}Pa'e^{-g} + Xg' = 0 \ , \tag{4.26}$$

$$-Pa - \frac{1}{2}a' + \frac{1}{2}(a^2 - 1)e^{-g}X = 0 \ , \tag{4.27}$$

where $X' = \partial_r X$ and $P' = \partial_r P$. The first two equations determine the remaining two unknown functions P and X. The rest of the equations are satisfied automatically by using the expressions for the functions a, g, Φ given in (4.3),(4.5).

To summarize, the *Kähler form* for the solution of [5, 10] is

$$\Omega = \xi^0 \wedge \rho^3 + X(r)(\rho^1 \wedge \xi^2 - \rho^2 \wedge \xi^1) + P(r)(-\rho^1 \wedge \rho^2 + \xi^1 \wedge \xi^2) \ , \tag{4.28}$$

where

$$P = \frac{\sinh 4r - 4r}{2\sinh^2 2r} \ , \qquad\qquad X = (1 - P^2)^{1/2} \ . \tag{4.29}$$

It remains to show that the almost complex structure associated with this Kähler form (4.28) is *integrable*. This can be verified by introducing the (1,0)-forms

$$\begin{aligned}
\mathcal{E}^1 &= \xi^0 + i\rho^3 \ , \\
\mathcal{E}^2 &= \rho^1 + iX\xi^2 - iP\rho^2 \ , \\
\mathcal{E}^3 &= \xi^1 + iX\rho^2 + iP\xi^2 \ .
\end{aligned} \tag{4.30}$$

Then according to the Frobenius theorem, the almost complex structure J is integrable provided that the (0,2) part of the 2-forms $d\mathcal{E}^1, d\mathcal{E}^2$ and $d\mathcal{E}^3$ vanishes. This requirement

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leads to the following conditions:

$$X + ae^{-g}P - \frac{1}{4}(a^2 - 1)e^{-2g}X = 0 ,$$

$$ae^{-g}X + X^2 g' + \frac{1}{2}e^{-g}PXa' = 0 ,$$

$$-X - ae^{-g}P - \frac{1}{4}e^{-g}a' - \frac{1}{2}PX'$$

$$+ \frac{1}{2}XP' - \frac{1}{2}XPg' - \frac{1}{4}e^{-g}P^2 a' = 0 ,$$

$$\frac{1}{2}g' + \frac{1}{2}ae^{-g}X + \frac{1}{4}e^{-g}PXa' - \frac{1}{2}P^2 g' = 0 ,$$

$$X + \frac{1}{2}PX' - \frac{1}{2}XP' + \frac{1}{4}e^{-g}X^2 a' - \frac{1}{2}PXg' = 0 , \tag{4.31}$$

which are satisfied automatically. Since $d\Omega + i_J H = 0$ is satisfied and the almost complex structure is integrable, then as it has been explained in Section 2, J is parallel with respect to the ∇^+ connection. Thus the first condition in (2.3) is satisfied.

It remains to prove that the second condition in (2.3) is satisfied as well. For this, it can be shown using various identities in [13] that if a background satisfies: (i) the field equations, (ii) admits a complex structure which is parallel with respect to the ∇^+ connection and (iii) obeys the Killing spinor equation associated with the dilatino, then the holonomy of the ∇^+ connection is contained in $SU(n)$. The NS5$_{S^2}$ background has been shown to obey all these three requirements and so it also satisfies the second condition (2.3). It follows from all the above that the NS5$_{S^2}$ background preserves at least four supersymmetries. It can be easily verified that the ∇^- connection does not admit a parallel complex structure with similar properties. Therefore, the holonomy of ∇^- is $SO(6)$. We conclude that the NS5$_{S^2}$ background preserves four supersymmetries (in agreement with the arguments in [5, 10]).

The ∇^+-parallel (3,0) form for the NS5$_{S^2}$ background is

$$\tilde{\eta} = \mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \mathcal{E}^3 , \tag{4.32}$$

and the associated holomorphic (3,0) form is

$$\eta = e^{-2\Phi}\mathcal{E}^1 \wedge \mathcal{E}^2 \wedge \mathcal{E}^3 . \tag{4.33}$$

Let us note that in the asymptotic limit $r \to \infty$, the NS5$_{S^2}$ solution simplifies considerably, with the functions in (4.3) becoming

$$\Phi = \Phi_0 - r + \frac{1}{4}\ln r , \qquad a = 0 , \qquad e^{2g} = r , \tag{4.34}$$

up to exponentially suppressed corrections. In this limit $P = 1$ and $X = 0$ (see (4.29)) so that the Kähler form (4.28) reduces to

$$\begin{aligned}
\Omega &= \xi^0 \wedge \rho^3 - \rho^1 \wedge \rho^2 + \xi^1 \wedge \xi^2 \\
&= \frac{1}{2}dr \wedge (\epsilon^3 - \cos\theta d\varphi) - \frac{1}{4}\epsilon^1 \wedge \epsilon^2 + r\sin\theta d\theta \wedge d\varphi .
\end{aligned} \tag{4.35}$$

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Finally, the ∇^+-parallel (3,0)-form is

$$\tilde{\eta} = \frac{\sqrt{r}}{2}[dr + \frac{i}{2}(\epsilon^3 - \cos\theta d\varphi)] \wedge (\epsilon^1 - i\epsilon^2) \wedge (d\theta + i\sin\theta d\varphi) , \qquad (4.36)$$

while the holomorphic (3,0) form is

$$\eta = \frac{1}{2}e^{2r}[dr + \frac{i}{2}(\epsilon^3 - \cos\theta d\varphi)] \wedge (\epsilon^1 - i\epsilon^2) \wedge (d\theta + i\sin\theta d\varphi) . \qquad (4.37)$$

5 Superpotential for 5-brane on S^2 solution

To complement the above discussion of complex geometry and supersymmetry of the NS5$_{S^2}$ background, let us now demonstrate that, like other $N = 1$ supersymmetric "D3-brane on conifold" type IIB solutions [3, 4, 8], this solution (in its S-dual D5$_{S^2}$ form [5]) can be derived from a first order system which follows from a superpotential. We shall also show that there exists a general ansatz for the background fields and the 1-d action for the radial evolution of which all of the solutions of [3, 4, 8] and [5] are special cases. That ansatz should be useful for finding more general solutions that "interpolate" between the "D3 + wrapped D5 on conifolds" and "D5 on S^2" backgrounds.

5.1 Interpolating ansatz for 3-branes on conifold and D5$_{S^2}$

The ansatz for the type IIB supergravity fields we shall choose below is motivated by the special cases studied in [3–5, 8]. We shall consider the 10-d metrics of topology $\mathbb{R}^{1,3} \times \mathbb{R} \times S^2 \times S^3$ with non-trivial functions depending only on the radial direction. The conifold metrics have $SU(2)_{L1} \times SU(2)_{L2}$ symmetry. The NS5$_{S^2}$ metric in (4.3) does not have the Z_2 symmetry between the two 2-spheres, so we are going to relax it, but we will insist on $SU(2)_{L1} \times U(1)_{L2}$ symmetry. We shall use the Einstein frame metric and embed the S-dual of the NS5$_{S^2}$ background, i.e. the 3-form in (4.3) will appear as special case of the R-R 3-form $H_{RR} \equiv F_3$.

As we will be dealing with 6-spaces of $\mathbb{R} \times S^2 \times S^3$ topology and metrics having $SU(2) \times U(1)$ isometry, let us define the relevant 1-forms: $\{e_1, e_2\}$ will correspond to the first S^2 (with coordinates θ_1, ϕ_1), and $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ will be the left-invariant forms on S^3 with Euler angle coordinates ψ, θ_2, ϕ_2 (cf. (3.8),(3.10),(4.6)):

$$e_1 \equiv d\theta_1 , \qquad e_2 \equiv -\sin\theta_1 d\phi_1 ,$$
$$\epsilon_1 \equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 , \qquad \epsilon_2 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 ,$$
$$\tilde{\epsilon}_3 = \epsilon_3 + \cos\theta_1 d\phi_1 \equiv (d\psi + \cos\theta_2 d\phi_2) + \cos\theta_1 d\phi_1 , \qquad (5.1)$$
$$d\epsilon_i = -\frac{1}{2}\epsilon_{ijk}\epsilon_j \wedge \epsilon_k .$$

Let us also introduce[7]

$$\tilde{\epsilon}_1 \equiv \epsilon_1 - a(u)e_1 , \qquad \tilde{\epsilon}_2 \equiv \epsilon_2 - a(u)e_2 , \tag{5.2}$$

where a is a function of the radial coordinate here denoted as u.

Our ansatz for the (Einstein-frame) 10-d metric which includes the metrics in [3, 4, 8] and [5] as special cases is $(m = 0, 1, 2, 3)$

$$ds_E^2 = e^{2p-x}(e^{2A}dx_m dx_m + du^2) + ds_5^2 , \tag{5.3}$$

$$ds_5^2 = e^{x+g}(e_1^2 + e_2^2) + e^{x-g}(\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2) + e^{-6p-x}\tilde{\epsilon}_3^2 \tag{5.4}$$

$$= (e^{x+g} + a^2 e^{x-g})(e_1^2 + e_2^2) + e^{x-g}[(\epsilon_1^2 + \epsilon_2^2) - 2a(\epsilon_1 e_1 + \epsilon_2 e_2)] + e^{-6p-x}\tilde{\epsilon}_3^2 , \tag{5.5}$$

where p, x, A, g, a are functions of u only. The function a thus multiplies the "off-diagonal" term $\epsilon_1 e_1 + \epsilon_2 e_2$. The Z_2 symmetry between the two spheres is broken unless $e^{x+g} + a^2 e^{x-g} = e^{x-g}$, i.e. $e^{2g} = 1 - a^2$ (note that $\epsilon_1^2 + \epsilon_2^2 \to e_1^2 + e_2^2$ under $(\theta_1, \phi_1) \to (\theta_2, \phi_2)$). In the singular and resolved conifold cases $a = 0$ [3, 8].

To specialize the above general ansatz to the case corresponding to the deformed conifold solution of [4] one is to relate g and a, replacing them by a single new function $y(u)$ as follows (see [8]):[8]

$$e^{-g} = \cosh y , \qquad a = \tanh y . \tag{5.6}$$

In the case of the D5-brane version of NS5$_{S^2}$ solution (4.1)–(4.3) [5]

$$A = \frac{2}{3}(g + \Phi) , \qquad x = g + \frac{1}{2}\Phi , \qquad p = -\frac{1}{6}(g + \Phi) ,$$

$$a = 2u(\sinh 2u)^{-1} , \qquad dr = e^{-\frac{2}{3}(g+\Phi)}du ,$$

$$e^{-2\Phi} = 2e^g(\sinh 2u)^{-1} , \qquad e^{2g} = u\coth 2u - \frac{1}{4}(1 + a^2) . \tag{5.7}$$

Here we use the Einstein-frame metric related to NS5$_{S^2}$ string-frame metric (4.5) by $ds_E^2 = e^{\Phi/2}(ds^2)_{NS5_{S^2}}$, where Φ now is the D5$_{S^2}$ dilaton, i.e. it has the opposite sign compared to the one in (4.3). To match the expressions in (4.5) one needs to make a rescaling $e^{\Phi/2} \to 4e^{\Phi/2}$, $e^{2g} \to \frac{1}{4}e^{2g}$, $r \to \frac{1}{2}r$ as here we did not include 1/4 factors in the S^3 part of the metric (5.4) (cf. (4.1)).

In addition to the dilaton $\Phi(u)$ we shall assume that only the two 3-form strengths and the 5-form field strengths of type IIB supergravity are non-zero. Our choice for these NS-NS and R-R forms will be

$$B_2 = h_1(u)(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + \chi(u)(-\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + h_2(u)(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1) ,$$

---

[7]To compare to the notation used in [4] (KS) and [5] (MN) note that: $e_1 = (e_2)_{KS} = (A_1/a)_{MN}$, $e_2 = (e_1)_{KS} = (A_2/a)_{MN}$, $\epsilon_1 = (e_4)_{KS} = (w_1)_{MN}$, $\epsilon_2 = (e_3)_{KS} = (w_2)_{MN}$, $\tilde{\epsilon}_3 = (e_5)_{KS} = (w_3 - A_3)_{MN}$, and $\tilde{\epsilon}_c = (w_c - A_c)_{MN}$. The forms used in KS in our notation are: $g^1 = -\frac{\epsilon_2 - e_2}{\sqrt{2}}$, $g^2 = -\frac{\epsilon_1 - e_1}{\sqrt{2}}$, $g^3 = \frac{\epsilon_2 + e_2}{\sqrt{2}}$, $g^4 = \frac{\epsilon_1 + e_1}{\sqrt{2}}$, $g^5 = \tilde{\epsilon}_3$. Note also that $\phi_1 = -(\phi)_{MN}$.

[8]The case of deformed conifold [4, 15, 16] corresponds to $a = -(\cosh\tau)^{-1}$, where τ is related to u so that the conifold metric takes the form $ds_6^2 = \frac{1}{2}\epsilon^{4/3}\mathcal{K}[(3\mathcal{K}^3)^{-1}(d\tau^2 + \tilde{\epsilon}_3^2) + \frac{1}{2}\sinh^2\tau(\cosh\tau)^{-1}(e_1^2 + e_2^2) + \frac{1}{2}\cosh\tau(\tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2)]$, where $\mathcal{K}^3 = (\frac{1}{2}\sinh 2\tau - \tau)/\sinh^3\tau$.

$$
\begin{aligned}
H_3 = dB_2 \; = \;\; & h_2(u)\tilde{\epsilon}_3 \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) + du \wedge [h_1'(u)(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) \\
+ \;\; & \chi'(u)(-\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + h_2'(u)(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)] \;,
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
F_3 = \; & P\tilde{\epsilon}_3 \wedge [\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b(u)(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)] \\
& + du \wedge [b'(u)(\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2)] \;,
\end{aligned}
\tag{5.9}
$$

$$
F_5 = \mathcal{F}_5 + \mathcal{F}_5^* \;, \qquad \mathcal{F}_5 = \mathrm{K}(u)e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \;.
\tag{5.10}
$$

Here $h_1, h_2, \chi, b, \mathrm{K}$ are functions of the radial direction u only (primes denote derivatives over u) and P is a constant. It is straightforward to check that the backgrounds in [4, 8] and [5] are special cases of this one. Note that, in general, the "off-diagonality" functions a and b in the metric (5.5) and in the R-R 3-form (5.9) are different (they are equal for the NS5$_{S^2}$ solution).

It is useful to write the forms (5.8),(5.9),(5.10) in the basis $\{e_a, \tilde{\epsilon}_i\}$ in which the metric (5.4) is diagonal:

$$
\begin{aligned}
H_3 = \; & h_2\tilde{\epsilon}_3 \wedge (\tilde{\epsilon}_1 \wedge e_1 + \tilde{\epsilon}_2 \wedge e_2) + du \wedge \Big[(h_1' - \chi')\tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 \\
& + [h_1'(1+a^2) + 2h_2'a + \chi'(1-a^2)]e_1 \wedge e_2 + (ah_1' + h_2' - a\chi')(\tilde{\epsilon}_1 \wedge e_2 - \tilde{\epsilon}_2 \wedge e_1)\Big] \;,
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
F_3 = \; & P\Big[\tilde{\epsilon}_3 \wedge [\tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 + (a^2 - 2ab + 1)e_1 \wedge e_2 \\
& + (a - b)(\tilde{\epsilon}_1 \wedge e_2 - \tilde{\epsilon}_2 \wedge e_1)] + du \wedge [b'(\tilde{\epsilon}_1 \wedge e_1 + \tilde{\epsilon}_2 \wedge e_2)]\Big] \;,
\end{aligned}
\tag{5.12}
$$

$$
\mathcal{F}_5 = \mathrm{K}(u)e_1 \wedge e_2 \wedge \tilde{\epsilon}_1 \wedge \tilde{\epsilon}_2 \wedge \tilde{\epsilon}_3 \;.
\tag{5.13}
$$

To obtain the 1-d action that leads to the equations of motion for all the unknown functions of u (i.e. p, x, A, g, a and $\Phi, h_1, h_2, \chi, b, \mathrm{K}$) one may follow the discussion in [3, 8] and use the equation for F_5 in combination with the relevant parts of the type IIB supergravity action

$$
S = \frac{1}{4} \int d^{10}x \sqrt{g_E} \Big[R - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{12}e^{-\Phi}H_3^2 - \frac{1}{12}e^{\Phi}F_3^2 - \frac{1}{4 \cdot 5!}F_5^2 \Big] + \mathrm{CS} - \mathrm{term} \;.
\tag{5.14}
$$

The effective 1-d action reproducing the equations of motion restricted to the above ansatz has the following general structure

$$
S_1 = \int du \; e^{4A}(3A'^2 + L) = \int du \; e^{4A}\Big[3A'^2 - \frac{1}{2}G_{ab}(\varphi)\varphi'^a\varphi'^b - V(\varphi)\Big] \;,
\tag{5.15}
$$

where φ^a stand for all unknown functions of u. This action should be supplemented with the "zero-energy" constraint $3A'^2 - \frac{1}{2}G_{ab}(\varphi)\varphi'^a\varphi'^b + V(\varphi) = 0$. Explicitly, using the metric in (5.5) one obtains

$$
\frac{1}{4} \int d^9x \sqrt{g_E} \; R \;\; \rightarrow \;\; e^{4A}(3A'^2 + L_{gr}) \;,
\tag{5.16}
$$

$$L_{gr} = -\frac{1}{2}x'^2 - \frac{1}{4}g'^2 - 3p'^2 - \frac{1}{4}e^{-2g}a'^2 - V_{gr} \ , \tag{5.17}$$

$$V_{gr} = -\frac{1}{2}e^{2p-2x}[e^g + (1+a^2)e^{-g}] + \frac{1}{8}e^{-4p-4x}[e^{2g} + (a^2-1)^2 e^{-2g} + 2a^2] + \frac{1}{4}a^2 e^{-2g+8p} \ . \tag{5.18}$$

The "matter" part of the 1-d action is found to be

$$L_m = -\frac{1}{8}\bigg[\Phi'^2 + e^{-\Phi-2x}\bigg(e^{2g}(h'_1 - \chi')^2$$

$$+ e^{-2g}[(1+a^2)h'_1 + 2ah'_2 + (1-a^2)\chi']^2 + 2(ah'_1 + h'_2 - a\chi')^2 + 2e^{8p}h_2^2\bigg)$$

$$+ P^2 e^{\Phi-2x}\bigg(e^{8p}[e^{2g} + e^{-2g}(a^2 - 2ab + 1)^2 + 2(a-b)^2] + 2b'^2\bigg)$$

$$+ e^{8p-4x}[Q + 2P(h_1 + bh_2)]^2\bigg] \ , \tag{5.19}$$

where we used that the type IIB supergravity equation for F_5 implies that

$$\mathrm{K}(u) = Q + 2P[h_1(u) + b(u)h_2(u)] \ , \qquad Q = \mathrm{const} \ . \tag{5.20}$$

Notice that χ enters the 1-d action (5.17),(5.19) only through its derivative, so that it can be eliminated using its equation of motion[9]

$$e^{2g}(h'_1 - \chi') + e^{-2g}(a^2 - 1)[(1+a^2)h'_1 + 2ah'_2 + (1-a^2)\chi']$$

$$+ 2a(ah'_1 + h'_2 - a\chi') = 0 \ . \tag{5.21}$$

Then we get

$$L_m = -\frac{1}{8}\bigg[\Phi'^2 + e^{-\Phi-2x}\bigg(2h_2'^2 + 4e^{-2g}(h'_1 + ah'_2)^2$$

$$- 4[e^{2g} + (1-a^2)^2 e^{-2g} + 2a^2]^{-1}[e^{-2g}(1-a^2)(h'_1 + ah'_2) - ah'_2]^2 + 2e^{8p}h_2^2\bigg)$$

$$+ P^2 e^{\Phi-2x}\bigg(e^{8p}[e^{2g} + e^{-2g}(a^2 - 2ab + 1)^2 + 2(a-b)^2] + 2b'^2\bigg)$$

$$+ e^{8p-4x}[Q + 2P(h_1 + bh_2)]^2\bigg] \ . \tag{5.22}$$

The cases considered in $[3, 4, 8]$ and $[5]$ are consistent truncations of this system (in particular, they are special solutions of the resulting system of equations):

(i) The fractional 3-brane on *singular* conifold solution [3] corresponds to $a = b = \chi = h_2 = g = 0$, $\Phi = \mathrm{const}$ (it is a special case of the deformed and resolved conifold backgrounds below).

(ii) The action corresponding to the 3-brane on *resolved* conifold solution [8] is found for $a = b = h_2 = 0$, $\Phi = \mathrm{const}$. In the notation of [8]

$$h_1 = \frac{1}{2}(f_1 - f_2) \ , \qquad \chi = \frac{1}{2}(f_1 + f_2) \ , \qquad g = y \ . \tag{5.23}$$

---

[9]The derivation and analysis of the next two equations was done in collaboration with S. Frolov.

(iii) The 1-d action corresponding to the 3-brane on *deformed* conifold case [4] is found [8] once one relates a and g according to (5.6) and also chooses $\Phi = $ const and $\chi = 0$ which is then a solution of (5.21). The relation to the functions used in [4, 8] is

$$h_1 = \frac{1}{2}(f+k) , \qquad h_2 = \frac{1}{2}(k-f) , \qquad b = P^{-1}F - 1 , \qquad a^2 = 1 - e^{2g} = \tanh^2 y . \quad (5.24)$$

(iv) The case of the $D5_{S^2}$ background [5] is $h_1 = h_2 = \chi = 0$, $b = a$. To satisfy the equations for a and b under the constraint $a = b$ one is also to impose the following relations

$$\Phi = -6p - g , \qquad x = g + \frac{1}{2}\Phi = \frac{1}{2}g - 3p , \qquad (5.25)$$

which lead precisely to the D5$_{S^2}$ ansatz (5.7).[10]

5.2   Superpotentials for special cases

In general, the existence of a superpotential means that V in (5.15) can be represented in the form

$$V = \frac{1}{8}G^{ab}\frac{\partial W}{\partial \varphi^a}\frac{\partial W}{\partial \varphi^b} - \frac{1}{3}W^2 . \qquad (5.26)$$

In this case the 2-nd order equations following from (5.15) and the "zero-energy" constraint are satisfied on the solutions of the 1-st order system

$$\varphi'^a = \frac{1}{2}G^{ab}\frac{\partial W}{\partial \varphi^b} , \qquad A' = -\frac{1}{3}W(\varphi) . \qquad (5.27)$$

In all of the four cases discussed above the 1-d system is such that it admits a simple superpotential W.

Indeed, in the resolved conifold case (5.23) we have $\varphi^a = (x, y, p, \Phi, f_1, f_2)$ and the 1-d Lagrangian (5.17),(5.18),(5.19) is

$$L = -\frac{1}{2}x'^2 - \frac{1}{4}y'^2 - 3p'^2 - \frac{1}{8}\left[\Phi'^2 + e^{-\Phi-2x}(e^{-2y}f_1'^2 + e^{2y}f_2'^2)\right] - V , \qquad (5.28)$$

$$V = \frac{1}{4}e^{-4p-4x}\cosh 2y - e^{2p-2x}\cosh y + \frac{1}{8}e^{8p}\left(2P^2e^{\Phi-2x}\cosh 2y + e^{-4x}[Q + P(f_1 - f_2)]^2\right) . \qquad (5.29)$$

The associated superpotential is [8]

$$W = e^{-2p-2x}\cosh y + e^{4p} + \frac{1}{2}e^{4p-2x}[Q + P(f_1 - f_2)] . \qquad (5.30)$$

The special "symmetric" solution $y = 0$ (implying also $f_1 = -f_2$, i.e. $\chi = 0$) corresponds to the singular conifold case [3].

---

[10] Again, to relate the variables here to the functions in (4.3) we need to change the sign of the dilaton and to rescale $e^{2g} \rightarrow \frac{1}{4}e^{2g}$.

In the deformed conifold case (5.24) [4] we have $\varphi^a = (x, y, p, \Phi, f, k, F)$ and the 1-d action (5.17),(5.18),(5.19) becomes

$$L = -\frac{1}{2}x'^2 - \frac{1}{4}y'^2 - 3p'^2 - \frac{1}{8}\left[\Phi'^2 + e^{-\Phi-2x}(e^{-2y}f'^2 + e^{2y}k'^2) + 2e^{\Phi-2x}F'^2\right] - V \ , \quad (5.31)$$

$$V = \frac{1}{4}e^{-4p-4x} - e^{2p-2x}\cosh y + \frac{1}{4}e^{8p}\sinh^2 y + \frac{1}{8}e^{8p}\left[\frac{1}{2}e^{-\Phi-2x}(f-k)^2\right.$$

$$\left. + e^{\Phi-2x}[e^{-2y}F^2 + e^{2y}(2P-F)^2] + e^{-4x}[Q + kF + f(2P-F)]^2\right] \ , \quad (5.32)$$

The corresponding superpotential [8] has the form similar to (5.30)

$$W = e^{-2p-2x} + e^{4p}\cosh y + \frac{1}{2}e^{4p-2x}[Q + kF + f(2P-F)] \ . \quad (5.33)$$

Note that, as in [3], W in these two cases happens to be dilaton-independent (so that $\Phi = $ const is a solution) and is simply a sum of a gravitational and matter parts.

In the D5$_{S^2}$ case the 1-d action takes the form (using $h_1 = h_2 = \chi = 0$, $b = a$ and (5.25))

$$L = -\frac{1}{2}g'^2 - \frac{1}{2}e^{-2g}a'^2 - 12p'^2 - V \ , \quad (5.34)$$

$$V = \frac{1}{4}e^{8p}\left[(a^2-1)^2e^{-4g} - 2e^{-2g} - 1\right] \ . \quad (5.35)$$

The superpotential for this system is quite different from (5.30),(5.33)[11]

$$W = e^{4p}\sqrt{(a^2-1)^2e^{-4g} + 2(a^2+1)e^{-2g} + 1} \ . \quad (5.36)$$

The first order system (5.27) following from (5.34),(5.36) is then solved by (5.7).

It would be very interesting to find other special cases of the above general ansatz that also admit superpotentials (and thus are likely to lead to new supersymmetric solutions). There must be at least one generalization of the wrapped 5-brane solution: it should be possible to extend the background of [5] away from the 5-brane throat region.[12]

Acknowledgments

---

[11]This difference should have to do with the fact that the dilaton is now non-constant and is mixed with the gravitational functions via (5.25). Note also that the kinetic term metric is no longer flat but is that of $AdS_2 \times \mathbb{R}$.

[12]To describe this case one needs to modify the original ansatz of [10] to include an extra scalar corresponding to the radius of S^3.

Appendix A Solution of $J^2 = -1$ condition for NS5$_{S^2}$

Here we solve the equations (4.14),(4.15),(4.16) arising from the condition $J^2 = -1$ on the complex structure. Let us determine $z_i{}^a$ in terms of 3-vectors λ_i and μ_i and 2-vector w_a. Solving the second equation in (4.15) we get (up or down position of indices does not matter as we use Euclidean signature)

$$z_i^a = \mathrm{z}_i^a + (P^2 - \mu^2)^{-1}\Big[(P\lambda_i - P^{-1}\mu \cdot \lambda \; \mu_i) \; \epsilon_{ab}w_b + \epsilon_{ijk}\lambda_j\mu_k \; w_a\Big] , \qquad (A.1)$$

where z_i^a is solution of homogeneous equation in (4.15). Then from (4.14) we find that

$$\lambda^2 = 1 - w^2 , \qquad \mu^2 = P^2 + w^2 , \qquad \mu \cdot \lambda \equiv \mu_i\lambda_i = P . \qquad (A.2)$$

Direct check shows that the first equation in (4.15) and (4.16) are satisfied identically. z_i^a is the general solution if either λ^i or w_a are equal to zero. Writing $\mathrm{z}_i^a = (x_i, y_i)$ we find from the second equation in (4.15)

$$Px_i = \epsilon_{ijk}\mu_j y_k , \qquad\qquad Py_i = -\epsilon_{ijk}\mu_j x_k , \qquad (A.3)$$

so that for $P \neq 0$

$$y \cdot \mu \; \mu_i = (\mu^2 - P^2)y_i , \qquad x \cdot \mu \; \mu_i = -(\mu^2 - P^2)x_i . \qquad (A.4)$$

From (4.14) we learn that for $w_a = 0$ we have λ_i as a unit 3-vector orthogonal to x_i and y_i, and parallel to μ_i. From (4.16) $x_i^2 = y_i^2 = 1 - P^2$. Thus we can choose $\mu^2 = P^2$, $\lambda_i = (0, 0, 1)$, $\mu_i = (0, 0, -P)$, and from (A.3)

$$x_i = (X, 0, 0) , \qquad y_i = (0, -X, 0) , \qquad X^2 = 1 - P^2 . \qquad (A.5)$$

This is the "radial" solution (4.19),(4.19),(4.21), with the asymptotic case (4.34),(4.35) being $X = 0$.

For $w \neq 0$ we can also write (A.1) as

$$z_i^a = \mathrm{z}_i^a - w^{-2}\Big[(P\lambda_i - \mu_i)\epsilon_{ab}w_b + \epsilon_{ijk}\lambda_j\mu_k w_a\Big] . \qquad (A.6)$$

Appendix B The Calabi metrics

Let us consider a metric is on the resolved $\mathbb{Z}_m$ singularity of $\mathbb{C}^m$. The Kähler potential is

$$K = (r^{2m} + 1)^{\frac{1}{m}} + \frac{1}{m}\sum_{j=0}^{m-1} \zeta^j \log\left[(r^{2m} + 1)^{\frac{1}{m}} - \zeta^j\right] , \qquad (B.1)$$

where $\zeta = e^{\frac{2\pi i}{m}}$ and $r^2 = \delta_{i\bar{j}}z^i\bar{z}^{\bar{j}}$ $(i, j = 1, \ldots, m)$. The coordinates z^i are those of $\mathbb{C}^m$. Observe that these metrics have a $U(m)$ isometry group.

There may be a generalization of these Calabi spaces [21] to include torsion. For this consider the ansatz for the metric

$$ds^2 = [\delta_{i\bar{j}}h(r^2) + \bar{z}_i z_{\bar{j}}k(r^2)]dz^i d\bar{z}^{\bar{j}} , \qquad (B.2)$$

where $\bar{z}_i = \delta_{i\bar{j}}\bar{z}^{\bar{j}}$ and $z_{\bar{i}} = \delta_{j\bar{i}}z^j$. It is well known that the associated torsion three-form H can be determined from the metric and the complex structure. In particular, we have

$$H = [\delta_{i\bar{j}}z_{\bar{k}}(h' - k) - \delta_{i\bar{k}}z_{\bar{j}}(h' - k)]dz^i \wedge d\bar{z}^{\bar{j}} \wedge d\bar{z}^{\bar{k}} + c.c. \ , \qquad (B.3)$$

where primes denote differentiation with respect to r^2. It remains to show that H is closed. This leads to the condition that

$$2(\delta_{k\bar{i}}\delta_{j\bar{\ell}} - \delta_{j\bar{i}}\delta_{k\bar{\ell}})(h' - k)\delta_{k\bar{i}}z_{\bar{\ell}}\bar{z}_j(h'' - k') -$$
$$\delta_{j\bar{i}}z_{\bar{\ell}}\bar{z}_k(h'' - k') - \delta_{k\bar{\ell}}z_{\bar{i}}\bar{z}_j(h'' - k') + \delta_{j\bar{\ell}}z_{\bar{i}}\bar{z}_k(h'' - k') \ = \ 0 \ . \qquad (B.4)$$

It turns out that if the dimension of the manifold is more than four, then the above equation has solutions provided that

$$h' - k = 0 \ , \qquad (B.5)$$

which implies that the torsion vanishes. So the only solution is that given by the Calabi metrics. In four dimensions, there is a solution with non-vanishing torsion that is represented by the NS5-brane background.

References

[1] J. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200]. S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. **B428** (1998) 105 [hep-th/9802109]. E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

[2] I.R. Klebanov and N.A. Nekrasov, "Gravity duals of fractional branes and logarithmic RG flow," Nucl. Phys. **B574** (2000) 263 [hep-th/9911096].

[3] I.R. Klebanov and A.A. Tseytlin, "Gravity duals of supersymmetric $SU(N) \times SU(N + M)$ gauge theories," Nucl. Phys. **B578** (2000) 123 [hep-th/0002159].

[4] I.R. Klebanov and M.J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and (chi)SB-resolution of naked singularities," hep-th/0007191.

[5] J.M. Maldacena and C. Nuñez, "Towards the large n limit of pure N = 1 super Yang Mills," hep-th/0008001.

[6] M. Graña and J. Polchinski, "Supersymmetric three-form flux perturbations on AdS(5)," hep-th/0009211.

[7] S.S. Gubser, "Supersymmetry and F-theory realization of the deformed conifold with three-form flux," hep-th/0010010.

[8] L.A. Pando Zayas and A.A. Tseytlin, "3-branes on resolved conifold," hep-th/0010088.

[9] M. Cvetic, H. Lu and C.N. Pope, "Brane resolution through transgression," hep-th/0011023.

[10] A.H. Chamseddine and M.S. Volkov, "Non-Abelian BPS monopoles in N = 4 gauged supergravity," Phys. Rev. Lett. **79**, 3343 (1997) [hep-th/9707176]. "Non-Abelian solitons in N = 4 gauged supergravity and leading order string theory," Phys. Rev. **D57**, 6242 (1998) [hep-th/9711181].

[11] A.M. Polyakov, "The wall of the cave," Int. J. Mod. Phys. **A14** (1999) 645 [hep-th/9809057]. "String theory and quark confinement," Nucl. Phys. Proc. Suppl. **68** (1998) 1 [hep-th/9711002].

[12] A. Strominger, "Superstrings With Torsion," Nucl. Phys. **B274**, 253 (1986).

[13] S. Ivanov and G. Papadopoulos, "A no-go theorem for string warped compactifications," hep-th/0008232. S. Ivanov and G. Papadopoulos, "Vanishing theorems and string backgrounds," math.dg/0010038.

[14] P. Candelas and X.C. de la Ossa, "Comments On Conifolds," Nucl. Phys. **B342** (1990) 246.

[15] R. Minasian and D. Tsimpis, "On the geometry of non-trivially embedded branes," Nucl. Phys. **B572** (2000) 499 [hep-th/9911042].

[16] K. Ohta and T. Yokono, "Deformation of conifold and intersecting branes," JHEP **0002** (2000) 023 [hep-th/9912266].

[17] I.R. Klebanov and E. Witten, "Superconformal field theory on threebranes at a Calabi-Yau singularity," Nucl. Phys. **B536** (1998) 199, [hep-th/9807080]. D.R. Morrison and M.R. Plesser, "Non-spherical horizons. I," Adv. Theor. Math. Phys. **3** (1999) 1 [hep-th/9810201]. I.R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," Nucl. Phys. **B556** (1999) 89 [hep-th/9905104].

[18] P.S. Howe and G. Papadopoulos, "Holonomy groups and W symmetries" Commun. Math. Phys. **151** (1993) 467 [hep-th/9202036].

[19] P.S. Howe and G. Papadopoulos, "Ultraviolet Behavior Of Two-Dimensional Supersymmetric Nonlinear Sigma Models," Nucl. Phys. **B289**, 264 (1987). "Further Remarks On The Geometry Of Two-Dimensional Nonlinear Sigma Models," Class. Quant. Grav. **5**, 1647 (1988). "Twistor spaces for HKT manifolds," Phys. Lett. **B379**, 80 (1996) [hep-th/9602108].

[20] G. Papadopoulos and A. Teschendorff, "Multiangle five-brane intersections", Phys. Lett. **B443** (1998) 159 [hep-th/9806191]; "Grassmannians, calibrations and five-brane intersections", Class. Quant. Grav. **17** (2000) 2641 [hep-th/9811034].

[21] E. Calabi "Métriques kählériennes et fibrés holomorhes", Annales scientifiques de l'École Normale Superrieure **12** (1979) 269.

[22] D.N. Page and C.N. Pope, "Which Compactifications Of D = 11 Supergravity Are Stable?," Phys. Lett. **B144** (1984) 346.